

On computing some special values of hypergeometric functions

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Abstract

The theoretical computing of special values assumed by the hypergeometric functions has a high interest not only on its own, but also in sight of the remarkable implications to both pure Mathematics and Mathematical Physics. Accordingly, in this paper we continue the path of research started in [19] and [20] providing some contribution to such functions computability inside and outside their disk of convergence. This is accomplished through two different approaches. The first is to provide some new results in the spirit of theorem 3.1 of [20], obtaining formulae for multivariable hypergeometric functions by generalizing a well known Kummer's identity [15] to the hypergeometric functions of two or more variable like those of Appell and Lauricella $F_D^{(n)}$.

In the second part, using some reduction schemes of hyperelliptic integrals due to Goursat [9] we evaluate Appell and Lauricella $F_D^{(n)}$ functions in some particular occurrences and in their analytic continuation.

KEYWORD: Reduction of Hyperelliptic Integrals; Complete Elliptic Integral of first kind; Hypergeometric Function; Appell Function; Lauricella Function

1 Introduction and aim of the paper

The theoretical evaluation of the hypergeometric functions has became of great consideration not only in order to provide a set of exact values for testing the numerical algorithms of their computation. But the best chiefly comes from being such aim accomplished by means of mathematical technics of remarkable power and interest. In fact it was by studying the hypergeometric series that two of the most important chapters of the functions theory were originated at the end of the XIX century, namely that on the linear differential equations (I. L. Fuchs), and the automorphic functions (H. Poincaré, F. Klein). Furthermore F. Klein has shown that from the hypergeometric equation just solved by the hypergeometric function, really comes out the majority of the linear differential equations of Mathematical Physics, if not all.

Just the uncountable applications of such functions to the Mathematical Physics have attired the numerical analysts in order to improve the quality, reliability and celerity of their effective computation. Coming to more recent developments, we can cite e.g. three papers by Joyce and Zucker, [13, 25, 14], where they employ for computational purpose the theory of singular elliptic moduli.

In order to provide some contribution to their computability, almost as a reference for some values of them inside and outside their disk of convergence, in this paper we continue the piece of research started in [19] and [20] following two aims. The first is to provide some new results in the spirit of theorem 3.1 of [20] where we proved that, if $2b > a > 0$

$${}_2F_1 \left(\begin{matrix} 2b-a; b \\ 2b \end{matrix} \middle| 2 \right) = (-i)^{2b-a} \sqrt{\pi} \frac{\Gamma \left(b + \frac{1}{2} \right)}{\Gamma \left(\frac{a+1}{2} \right) \Gamma \left(\frac{2b-a+1}{2} \right)}$$

What above follows from the *double evaluation method* (namely in terms of Gamma function and using the integral representation theorem, see (2.2) below) applied to the improper integral

$$\int_0^\infty \frac{t^{a-1}}{(1+t^2)^b} dt.$$

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Similarly, here we obtain formulae involving hypergeometric functions of two or more variables (Appell F_1 and Lauricella $F_D^{(n)}$) using two sequences of integrals

$$A_n(a, b) = \int_0^1 \frac{t^{a-1}}{(1-t^n)^b} dt, \quad B_n(a, b) = \int_0^\infty \frac{t^{a-1}}{(1+t^n)^b} dt.$$

In this way we succeed in generalizing the Kummer's identity (see [15], [3] pp. 9-10, or Cor. 3.1.2 p. 126 in [1]):

$${}_2F_1 \left(\begin{matrix} a; b \\ 1+a-b \end{matrix} \middle| -1 \right) = \frac{\Gamma(1+a-b) \Gamma(1+\frac{a}{2})}{\Gamma(1+a) \Gamma(1+\frac{a}{2}-b)} \quad (1.1)$$

concerning the Gauss function, to hypergeometric functions of two or more variable like those of Appell and Lauricella $F_D^{(n)}$.

In the second part, using some reduction of a class of hyperelliptic integrals due to Goursat [9] we evaluate Appell and Lauricella $F_D^{(n)}$ in some particular occurrences and in their analytic continuation. This continues our previous research of [19] and [20] where we obtained several identities connecting π , elliptic integrals and hypergeometric function using some reduction schemes of hyperelliptic integrals due to Jacobi [12] and to Hermite [11].

2 Notations

In this paper whenever ${}_2F_1$ denotes a well-known hypergeometric series:

$${}_2F_1 \left(\begin{matrix} a; b \\ c \end{matrix} \middle| x \right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}, \quad (2.1)$$

it also denotes its analytic continuation on $\mathbb{C} \setminus [1, \infty)$ via its integral representation theorem $\operatorname{Re}(a) > 0$, $\operatorname{Re}(c-a) > 0$, $|x| < 1$:

$${}_2F_1 \left(\begin{matrix} a; b \\ c \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 \frac{u^{a-1} (1-u)^{c-a-1}}{(1-xu)^b} du \quad (2.2)$$

In (2.1) $(a)_m$ is a Pochhammer symbol:

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1) \cdots (a+m-1).$$

We also consider the multivariable extensions of ${}_2F_1$: the Appell F_1 two-variables hypergeometric series, see [2], defined for $|x_1| < 1$, $|x_2| < 1$:

$$F_1 \left(\begin{matrix} a; b_1, b_2 \\ c \end{matrix} \middle| x_1, x_2 \right) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2}}{(c)_{m_1+m_2}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!},$$

the analytic continuation for the Appell's function on $\mathbb{C} \setminus [1, \infty) \times \mathbb{C} \setminus [1, \infty)$ comes from its integral representation theorem: if $\operatorname{Re}(a) > 0$, $\operatorname{Re}(c-a) > 0$:

$$F_1 \left(\begin{matrix} a; b_1, b_2 \\ c \end{matrix} \middle| x_1, x_2 \right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{u^{a-1} (1-u)^{c-a-1}}{(1-x_1 u)^{b_1} (1-x_2 u)^{b_2}} du.$$

Its n -variable extension is the Lauricella hypergeometric series, introduced in [16]:

$$F_D^{(n)} \left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix} \middle| x_1, \dots, x_n \right) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}$$

whose integral representation theorem for $\operatorname{Re}(a) > 0$, $\operatorname{Re}(c-a) > 0$ allows its analytic continuation on the n -fold cartesian product of $\mathbb{C} \setminus [1, \infty)^n$

$$F_D^{(n)} \left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix} \middle| x_1, \dots, x_n \right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{u^{a-1} (1-u)^{c-a-1}}{(1-x_1 u)^{b_1} \cdots (1-x_n u)^{b_n}} du. \quad (2.3)$$

We will also use the order reduction formula for Lauricella functions

$$F_D^{(n)} \left(\begin{matrix} a; b_1, \dots, b_n \\ b_1 + \dots + b_n \end{matrix} \middle| x_1, \dots, x_n \right) = \frac{1}{(1-x_n)^a} F_D^{(n-1)} \left(\begin{matrix} a, b_1, \dots, b_{n-1} \\ b_1 + \dots + b_{n-1} \end{matrix} \middle| \frac{x_1 - x_n}{1 - x_n}, \dots, \frac{x_{n-1} - x_n}{1 - x_n} \right) \quad (2.4)$$

whose proof was given in [20] Lemma 1.1 therein.

3 Special values from Eulerian integrals

3.1 Evaluation in the boundary of the unit disk

We will present here a generalization of the Kummer identity to both Appell and Lauricella functions. Our starting point is an elementary proof of (1.1) we discovered, at least we presume, throughout a two-fold evaluation of the definite integral:

$$A_2(a, b) = \int_0^1 \frac{t^{a-1}}{(1-t^2)^b} dt \quad (3.1)$$

where, for the sake of convergence of (3.1), we assume $\text{Re}(a) > 0$ and $\text{Re}(b) < 1$. First we change variable in (3.1) putting $t^2 = u$; it follows that

$$A_2(a, b) = \frac{1}{2} \frac{\Gamma(\frac{a}{2}) \Gamma(1-b)}{\Gamma(1 + \frac{a}{2} - b)}.$$

On the other side, if we write (3.1) as

$$A_2(a, b) = \int_0^1 \frac{t^{a-1}(1-t)^{-b}}{(1+t)^b} dt. \quad (3.2)$$

From the integral representation theorem, equation (2.2), we get:

$$A_2(a, b) = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1-a-b)} {}_2F_1 \left(\begin{matrix} a; b \\ 1+a-b \end{matrix} \middle| -1 \right). \quad (3.3)$$

By comparing both expressions for $A_2(a, b)$ and recalling that

$$\frac{\Gamma(\frac{a}{2})}{2\Gamma(a)} = \frac{\Gamma(1 + \frac{a}{2})}{\Gamma(1+a)}$$

we infer (1.1) which was originally given by Kummer in [15].

To obtain for the Appell F_1 two variables hypergeometric function something analogous to formula (1.1), we consider the integral, where we changed the exponent 2 in (3.1) to 3:

$$A_3(a, b) = \int_0^1 \frac{t^{a-1}}{(1-t^3)^b} dt. \quad (3.4)$$

Again putting $t^3 = u$ we evaluate (3.4) through the Eulerian integral of the first kind as

$$A_3(a, b) = \frac{1}{3} \frac{\Gamma(\frac{a}{3}) \Gamma(1-b)}{\Gamma(1 + \frac{a}{3} - b)}.$$

On the other side we can factorize $1 - t^3$ obtaining

$$A_3(a, b) = \int_0^1 \frac{t^{a-1}(1-t)^{-b}}{(1 - e^{\frac{2}{3}\pi i} t)^b (1 - e^{-\frac{2}{3}\pi i} t)^b} dt = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} F_1 \left(\begin{matrix} a; b, b \\ 1+a-b \end{matrix} \middle| e^{\frac{2}{3}\pi i}, e^{-\frac{2}{3}\pi i} \right).$$

Comparing the expression for $A_3(a, b)$ we get

$$F_1 \left(\begin{matrix} a; b, b \\ 1+a-b \end{matrix} \middle| e^{\frac{2}{3}\pi i}, e^{-\frac{2}{3}\pi i} \right) = \frac{\Gamma(1+a-b) \Gamma(1 + \frac{a}{3})}{\Gamma(1+a) \Gamma(1 + \frac{a}{3} - b)}. \quad (3.5)$$

Notice that we use the trivial identity, $n \in \mathbb{N}$

$$\frac{\Gamma\left(\frac{a}{n}\right)}{n\Gamma(a)} = \frac{\Gamma\left(1 + \frac{a}{n}\right)}{\Gamma(1+a)}$$

taking $n = 3$.

Continuing, from the integral

$$\int_0^1 \frac{t^{a-1}}{(1-t^4)^b} dt = \frac{\Gamma\left(\frac{a}{4}\right)\Gamma(1-b)}{4\Gamma\left(1 + \frac{a}{4} - b\right)} \quad (3.6)$$

factorizing $1 - t^4$ we get the Kummer-like formula for the Lauricella $F_D^{(3)}$

$$F_D^{(3)} \left(\begin{matrix} a; b, b, b \\ 1 + a - b \end{matrix} \middle| -1, i, -i \right) = \frac{\Gamma(a-b+1)\Gamma\left(1 + \frac{a}{4}\right)}{\Gamma(1+a)\Gamma\left(1 + \frac{a}{4} - b\right)} \quad (3.7)$$

We see that the general relation for $n \in \mathbb{N}$, $n \geq 2$ is

$$A_n(a, b) = \frac{\Gamma\left(\frac{a}{n}\right)\Gamma(1-b)}{n\Gamma\left(1 + \frac{a}{n} + b\right)}. \quad (3.8)$$

Thus we can state our result in general. In order to simplify the notation whenever in a Lauricella function the n parameters as b are all equal, $b_1 = \dots = b_n = b$, we put

$$F_D^{(n)} \left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix} \middle| x_1, \dots, x_n \right) = F_D^{(n)} \left(\begin{matrix} a; b \\ c \end{matrix} \middle| x_1, \dots, x_n \right)$$

the repetitions number of b is denoted by the apex of the Lauricella function.

Theorem 3.1. *Let n be a positive integer ≥ 2 . For $k = 1, \dots, n-1$ let $\omega_{k,n} = e^{i\frac{2k\pi}{n}}$ so that $\omega_{k,n}^n = 1$ and $\omega_{k,n} \neq 1$ for $k = 1, \dots, n-1$. Let $0 < \text{Re}(b) < 1$ and $\text{Re}(a) > 0$ then*

$$F_D^{(n-1)} \left(\begin{matrix} a, b \\ 1 + a - b \end{matrix} \middle| \omega_{1,n}, \dots, \omega_{n-1,n} \right) = \frac{\Gamma(a-b+1)\Gamma\left(1 + \frac{a}{n}\right)}{\Gamma(1+a)\Gamma\left(1 + \frac{a}{n} - b\right)} \quad (3.9)$$

Proof. The statement follows from the integral representation theorem (2.3) from which we can establish relation (3.10) below that expresses integrals $A_n(a, b)$ in terms of $n-1$ variables Lauricella function $F_D^{(n-1)}$

$$A_n(a, b) = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} F_D^{(n-1)} \left(\begin{matrix} a, b \\ 1 + a - b \end{matrix} \middle| \omega_{1,n}, \dots, \omega_{n-1,n} \right) \quad (3.10)$$

This follows comparing (3.10) with (3.8). □

Remark 3.2. In the case $n = 2$ when $a = 1$ and $b = 1/2$ we have the elementary value $A_2(1, 1/2) = \pi/2$ which drives to

$${}_2F_1 \left(\begin{matrix} 1, 1/2 \\ 3/2 \end{matrix} \middle| -1 \right) = \frac{\pi}{4}$$

according to equation (14) section 2.8 p. 102 of [7]:

$$\frac{\arctan z}{z} = {}_2F_1 \left(\begin{matrix} 1, 1/2 \\ 3/2 \end{matrix} \middle| -z^2 \right).$$

3.2 Values outside the unit disk

Consider the sequence of integrals

$$B_n(a, b) = \int_0^\infty \frac{t^{a-1}}{(1+t^n)^b} dt \quad (3.11)$$

where we assume $a > 0$, $b > 0$, $na > b$ to ensure convergence. The integral at the right hand side of (3.11) can be computed by means of the Mellin transform in [8] p. 310 equation 21, see also [10] entry 3.194-3 p. 313, so that

$$B_n(a, b) = \frac{\Gamma(\frac{a}{n})\Gamma(\frac{nb-a}{n})}{n\Gamma(b)}.$$

On the other side, we can also evaluate $B_n(a, b)$ using the integral representation theorem (2.3). In fact change of variable $t = (1-u)/u$ drives to

$$B_n(a, b) = \int_0^1 \frac{u^{nb-a-1}(1-u)^{a-1}}{(u^n + (1-u)^n)^b} du. \quad (3.12)$$

Observe that for n even the polynomial $u^n + (1-u)^n$ has degree n , while for n odd has degree $n-1$. In any case the roots are distinct and the factorization holds:

$$u^n + (1-u)^n = \prod_k \left(1 - \frac{1}{\alpha_k} u\right)$$

where α_k is root of $u^n + (1-u)^n = 0$. These roots are evaluated, through De Moivre formula, as follows. If n is even, say $n = 2m$, we have

$$\alpha_k = \frac{1}{2} - i \frac{\sin \frac{(2k-1)\pi}{2m}}{2 \left(1 + \cos \frac{(2k-1)\pi}{2m}\right)}, \quad k = 1, \dots, 2m. \quad (3.13)$$

If n is odd, say $n = 2m-1$

$$\alpha_k = \frac{1}{2} - i \frac{\sin \frac{(2k-1)\pi}{2m-1}}{2 \left(1 + \cos \frac{(2k-1)\pi}{2m-1}\right)}, \quad k = 1, \dots, 2m-1, k \neq m. \quad (3.14)$$

We are in position to state our second result.

Theorem 3.3. *Assume that $a > 0$, $b > 0$, $nb > a$. If $n = 2m$ is even then*

$$F_D^{(2m)} \left(\begin{matrix} 2mb-a; b \\ 2mb \end{matrix} \middle| x_1, \dots, x_{2m} \right) = \frac{1}{2m} \frac{\Gamma(\frac{a}{2m})\Gamma(2mb)\Gamma(\frac{2mb-a}{2m})}{\Gamma(a)\Gamma(b)\Gamma(2mb-a)} \quad (3.15)$$

where, from (3.15) for $k = 1, \dots, 2m$:

$$x_k = 1 + \cos \frac{(2k-1)\pi}{2m} + i \sin \frac{(2k-1)\pi}{2m}. \quad (3.16)$$

If $n = 2m-1$ is odd then

$$F_D^{(2m-2)} \left(\begin{matrix} (2m-1)b-a; b \\ (2m-1)b \end{matrix} \middle| y_1, \dots, y_{2m-1} \right) = \frac{1}{2m-1} \frac{\Gamma(\frac{a}{2m-1})\Gamma((2m-1)b)\Gamma(\frac{(2m-1)b-a}{2m-1})}{\Gamma(a)\Gamma(b)\Gamma((2m-1)b-a)} \quad (3.17)$$

where from (3.17) for $k = 1, \dots, 2m-1$, $k \neq m$

$$y_k = 1 + \cos \frac{(2k-1)\pi}{2m-1} + i \sin \frac{(2k-1)\pi}{2m-1}.$$

Remark 3.4. In the even case $n = 2m$ it is possible, using Lemma 1.1 of [20], reduce the order of Lauricella function appearing in (3.15), obtaining:

$$\left(-e^{\frac{i\pi}{2m}}\right)^{2bm-a} F_D^{(2m-1)} \left(\begin{matrix} 2mb-a; b \\ 2mb \end{matrix} \middle| z_1, \dots, z_{2m-1} \right) = \frac{1}{2m} \frac{\Gamma(\frac{a}{2m})\Gamma(2mb)\Gamma(\frac{2mb-a}{2m})}{\Gamma(a)\Gamma(b)\Gamma(2mb-a)}$$

with arguments given for $k = 1, \dots, 2m-1$ by

$$z_k = 1 - \cos \frac{\pi k}{m} - i \sin \frac{\pi k}{m}.$$

Thus for $m = 1$ we obtain, as a particular case, theorem 3.1 of [20].

Proof. Using integral representation theorem, formula (2.3) in equation (3.12) we get, if $n = 2m$ is even

$$B_n(a, b) = \frac{\Gamma(a)\Gamma(nb-a)}{\Gamma(nb)} F_D^{(n)} \left(\begin{matrix} nb-a; b \\ nb \end{matrix} \middle| x_1, \dots, x_n \right) \quad (3.18)$$

where we denote with x_1, \dots, x_n the reciprocal of the n roots of equation $u^n + (1-u)^n = 0$ given by (3.16). While if $n = 2m-1$ is odd we have

$$B_n(a, b) = \frac{\Gamma(a)\Gamma(nb-a)}{\Gamma(nb)} F_D^{(n-1)} \left(\begin{matrix} nb-a; b \\ nb \end{matrix} \middle| x_1, \dots, x_n \right) \quad (3.19)$$

but here root $x_{\frac{n+1}{2}} = x_m$ is skipped. Theses (3.15) and (3.17) follow comparing respectively (3.18) and (3.19) with (3.11). \square

4 Some special cases

When in some special situation the Eulerian integrals $A_n(a, b)$ and $B_n(a, b)$ can be evaluated in terms of complete elliptic integral of first and second kind we are able to establish, from our formulae (3.9) and (3.17), new relationships between hypergeometric functions of one or more variable at some special values and certain complete elliptic integrals. This relationships arise for singular moduli of the elliptic integrals, according to the relevant theory studied by several authors, see [22, 24, 4, 5]. We will use some integrals taken from [10] and some less known integral studied in the past by some eminent mathematicians of XIX century, Legendre [17] and [18], Richelot [21], Serret [23], integrals, as far as we know, almost forgotten by the recent literature.

Starting with the most elementary elliptic integrals, notice that in any occurrence we also express the integral under investigation in terms of our families $A_n(a, b)$ or $B_n(a, b)$

$$B_4(1, \frac{1}{2}) = \int_0^\infty \frac{dx}{\sqrt{1+x^4}} = K\left(\frac{1}{\sqrt{2}}\right) \quad (4.1)$$

$$B_3(\frac{1}{2}, \frac{1}{2}) = \int_0^\infty \frac{dx}{\sqrt{x(1+x^3)}} = K\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) \quad (4.2)$$

Using (3.18) and (3.19) we see that

$$F_D^{(4)} \left(\begin{matrix} 1; \frac{1}{2} \\ 2 \end{matrix} \middle| 1 + \frac{1+i}{\sqrt{2}}, 1 - \frac{1-i}{\sqrt{2}}, 1 - \frac{1+i}{\sqrt{2}}, 1 + \frac{1-i}{\sqrt{2}} \right) = K\left(\frac{1}{\sqrt{2}}\right) \quad (4.3)$$

$$F_1 \left(\begin{matrix} 1; \frac{1}{2}, \frac{1}{2} \\ 3/2 \end{matrix} \middle| \frac{3}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2} \right) = \frac{2}{\sqrt[4]{27}} K\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) \quad (4.4)$$

Using Lemma 1.1 of [20] in (4.3) we also establish values of $F_D^{(3)}$ when one of the variables is in the positive real axis in analogy with our results for the Gauss ${}_2F_1$ exposed in [20] theorem 3.2 equations (27) to (30) therein

$$F_D^{(3)} \left(\begin{matrix} 1; \frac{1}{2} \\ 2 \end{matrix} \middle| 1-i, 2, 1+i \right) = -\frac{1-i}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right)$$

Now we turn our attention to some integrals from classical repertories [10] and [6]. We begin with integral 3.183.2 p. 313 of [10]

$$A_2(1, \frac{1}{4}) = \int_0^1 \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2} \left[2E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right) \right]$$

so that comparing with (3.3) we get the formula

$${}_2F_1\left(\begin{matrix} 1; \frac{1}{4} \\ \frac{7}{4} \end{matrix} \middle| -1\right) = \frac{3}{4}\sqrt{2} \left[2E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right) \right] \quad (4.5)$$

From entry 3.184.1 p. 314 of [10]

$$A_2(3, \frac{1}{4}) = \int_0^1 \frac{x^2}{\sqrt[4]{1-x^2}} dx = \frac{2\sqrt{2}}{5} \left[2E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right) \right]$$

we obtain

$${}_2F_1\left(\begin{matrix} 3; \frac{1}{4} \\ \frac{15}{4} \end{matrix} \middle| -1\right) = \frac{231\sqrt{2}}{320} \left[2E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right) \right] \quad (4.6)$$

Using formula 3.185.2 p. 314 of [10]

$$A_2(1, \frac{3}{4}) = \int_0^1 \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2} K\left(\frac{1}{\sqrt{2}}\right)$$

we infer

$${}_2F_1\left(\begin{matrix} 1; \frac{3}{4} \\ \frac{5}{4} \end{matrix} \middle| -1\right) = \frac{\sqrt{2}}{4} K\left(\frac{1}{\sqrt{2}}\right) \quad (4.7)$$

We end the examples taken from [10] with entry 3.185.4 p. 314

$$A_2(3, \frac{3}{4}) = \int_0^1 \frac{x^2}{\sqrt[4]{(1-x^2)^3}} dx = \frac{2\sqrt{2}}{3} K\left(\frac{1}{\sqrt{2}}\right)$$

which drives to

$${}_2F_1\left(\begin{matrix} 3; \frac{3}{4} \\ \frac{13}{4} \end{matrix} \middle| -1\right) = \frac{15}{32\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right) \quad (4.8)$$

Consider entries 576.00 p. 256 and 578.00 p. 258 of [6]

$$\int_0^1 \frac{dx}{\sqrt{1-x^6}} = \frac{1}{\sqrt[4]{3}} K\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) \quad (4.9)$$

$$\int_0^\infty \frac{dx}{\sqrt{1+x^6}} = \frac{2}{\sqrt[4]{27}} K\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) \quad (4.10)$$

Comparing (4.9) with (3.10) and (4.10) with (3.18) we obtain

$$F_D^{(5)}\left(\begin{matrix} 1; \frac{1}{2} \\ \frac{3}{2} \end{matrix} \middle| x_1^{(6)}, x_2^{(6)}, x_3^{(6)}, x_4^{(6)}, x_5^{(6)}\right) = \frac{1}{4\sqrt[4]{3}} K\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) \quad (4.11)$$

$$F_D^{(6)}\left(\begin{matrix} 2; \frac{1}{2} \\ 3 \end{matrix} \middle| y_1^{(6)}, y_2^{(6)}, y_3^{(6)}, y_4^{(6)}, y_5^{(6)}, y_6^{(6)}\right) = \frac{4}{\sqrt[4]{27}} K\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) \quad (4.12)$$

where

$$x_1^{(6)} = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad x_2^{(6)} = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad x_3^{(6)} = -1, \quad x_4^{(6)} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad x_5^{(6)} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

and

$$\begin{aligned} y_1^{(6)} &= 1 + \frac{\sqrt{3}}{2} + \frac{i}{2}, & y_2^{(6)} &= 1 + i, & y_3^{(6)} &= 1 - \frac{\sqrt{3}}{2} + \frac{i}{2} \\ y_4^{(6)} &= 1 - \frac{\sqrt{3}}{2} - \frac{i}{2}, & y_5^{(6)} &= 1 - i, & y_6^{(6)} &= 1 + \frac{\sqrt{3}}{2} - \frac{i}{2} \end{aligned}$$

Observe that employing (3.19) in (4.12) we evaluate $F_D^{(5)}$ when one of its arguments is 2. Putting

$$z_1^{(6)} = \frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad z_2^{(6)} = \frac{3}{2} - \frac{i\sqrt{3}}{2}, \quad z_3^{(6)} = 2, \quad z_4^{(6)} = \frac{3}{2} + \frac{i\sqrt{3}}{2}, \quad z_5^{(6)} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

we have

$$F_D^{(5)} \left(\begin{array}{c} 2; \frac{1}{2} \\ 3 \end{array} \middle| z_1^{(6)}, z_2^{(6)}, z_3^{(6)}, z_4^{(6)}, z_5^{(6)} \right) = \frac{4}{\sqrt[4]{27}} \left(-\frac{\sqrt{3}}{2} + \frac{i}{2} \right) \mathbf{K} \left(\frac{\sqrt{6} - \sqrt{2}}{4} \right) \quad (4.13)$$

Now we consider some integrals studied by Legendre, Richelot and Serret. Legendre, [18] p. 383, proved that

$$A_8(1, \frac{1}{2}) = \int_0^1 \frac{dx}{\sqrt{1-x^8}} = \frac{1}{\sqrt{2}} \mathbf{K}(\sqrt{2}-1) \quad (4.14)$$

thus comparing (4.14) with (3.10) we get

$$F_D^{(7)} \left(\begin{array}{c} 1; \frac{1}{2} \\ \frac{3}{2} \end{array} \middle| \frac{1+i}{\sqrt{2}}, i, -\frac{1-i}{\sqrt{2}}, -1, -\frac{1+i}{\sqrt{2}}, -i, \frac{1-i}{\sqrt{2}} \right) = \frac{1}{2\sqrt{2}} \mathbf{K}(\sqrt{2}-1) \quad (4.15)$$

Richelot in [21] evaluated

$$A_8(3, \frac{1}{2}) = \int_0^1 \frac{x^2}{\sqrt{1-x^8}} dx = \left(1 - \frac{1}{\sqrt{2}}\right) \mathbf{K}(\sqrt{2}-1) \quad (4.16)$$

thus comparing (4.16) with (3.10) we obtain

$$F_D^{(7)} \left(\begin{array}{c} 3; \frac{1}{2} \\ \frac{7}{2} \end{array} \middle| \frac{1+i}{\sqrt{2}}, i, -\frac{1-i}{\sqrt{2}}, -1, -\frac{1+i}{\sqrt{2}}, -i, \frac{1-i}{\sqrt{2}} \right) = \frac{15}{16} \left(1 - \frac{1}{2\sqrt{2}}\right) \mathbf{K}(\sqrt{2}-1) \quad (4.17)$$

To establish further identities, we recall two huge formulae due to Legendre, that, as far as we know, are quite unrecognized by the mathematical community of our age. In his famous *Traité*, [18] p. 377, equations (z) therein, he established two remarkable formulae, the first, for $2a < n$

$$\int_0^1 \frac{x^{a-1}}{\sqrt{1-x^n}} dx = \cos\left(\frac{a}{n}\pi\right) \int_0^\infty \frac{z^{a-1}}{\sqrt{1+z^n}} dz \quad (4.18)$$

the second, for $n/2 < a < n$, is

$$\int_0^\infty \frac{z^{n-a-1}}{\sqrt{1+z^n}} dz \cdot \int_0^1 \frac{x^{a-1}}{\sqrt{1-x^n}} dx = \frac{2\pi}{n(2a-n)\sin(\frac{\pi}{n}a)}. \quad (4.19)$$

We do not want to report here the path followed by Legendre to establish (4.18) and (4.19), a very interesting topic, with the original source now easily accessible. We highlight that (4.18) plays a role in order to evaluate the integral (4.14) so that it can be used also for evaluating the integral (4.16).

Now by formulae (4.18) and (4.19) let us pass to some evaluation of elliptic integrals related to (4.14) and (4.16):

$$B_8(1, \frac{1}{2}) = \int_0^\infty \frac{dx}{\sqrt{1+x^8}} = \sqrt{2-\sqrt{2}} \mathbf{K}(\sqrt{2}-1) \quad (4.20)$$

$$B_8(3, \frac{1}{2}) = \int_0^\infty \frac{x^2}{\sqrt{1+x^8}} dx = \sqrt{2-\sqrt{2}} \mathbf{K}(\sqrt{2}-1) \quad (4.21)$$

$$A_8(5, \frac{1}{2}) = \int_0^1 \frac{x^4}{\sqrt{1-x^8}} dx = \frac{\pi}{8} \frac{\sqrt{2}}{\mathbf{K}(\sqrt{2}-1)} \quad (4.22)$$

$$A_8(7, \frac{1}{2}) = \int_0^1 \frac{x^6}{\sqrt{1-x^8}} dx = \frac{\pi}{24} \frac{2+\sqrt{2}}{\mathbf{K}(\sqrt{2}-1)} \quad (4.23)$$

Thus taking advantage of (3.15) in (4.20) and (4.21) we see that

$$F_D^{(8)} \left(\begin{matrix} 3; \frac{1}{2} \\ 4 \end{matrix} \middle| x_1, \dots, x_8 \right) = 3\sqrt{2-\sqrt{2}} \mathbf{K}(\sqrt{2}-1) \quad (4.24)$$

$$F_D^{(8)} \left(\begin{matrix} 1; \frac{1}{2} \\ 4 \end{matrix} \middle| x_1, \dots, x_8 \right) = 3\sqrt{2-\sqrt{2}} \mathbf{K}(\sqrt{2}-1) \quad (4.25)$$

with x_1, \dots, x_8 reciprocal of the roots of $u^8 + (1-u)^8 = 0$ as given by (3.18), namely:

$$\begin{aligned} x_1 &= 1 + \frac{1}{2}i\sqrt{2-\sqrt{2}} + \frac{\sqrt{2+\sqrt{2}}}{2}, & x_2 &= 1 + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{1}{2}i\sqrt{2+\sqrt{2}} \\ x_3 &= 1 - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{1}{2}i\sqrt{2+\sqrt{2}}, & x_4 &= 1 + \frac{1}{2}i\sqrt{2-\sqrt{2}} - \frac{\sqrt{2+\sqrt{2}}}{2}, \\ x_5 &= 1 - \frac{1}{2}i\sqrt{2-\sqrt{2}} - \frac{\sqrt{2+\sqrt{2}}}{2}, & x_6 &= 1 - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{1}{2}i\sqrt{2+\sqrt{2}} \\ x_7 &= 1 + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{1}{2}i\sqrt{2+\sqrt{2}}, & x_8 &= 1 - \frac{1}{2}i\sqrt{2-\sqrt{2}} + \frac{\sqrt{2+\sqrt{2}}}{2} \end{aligned}$$

We can also reduce the order of the Lauricella functions in (4.24) and (4.25) using Lemma 1.1 of [20] getting special values of $F_D^{(7)}$ with some of the arguments lying in the positive real axis

$$F_D^{(7)} \left(\begin{matrix} 3; \frac{1}{2} \\ 4 \end{matrix} \middle| z_1, \dots, z_7 \right) = \left(-\frac{3}{\sqrt{2}} + i \left(3 - \frac{3}{\sqrt{2}} \right) \right) \mathbf{K}(\sqrt{2}-1) \quad (4.26)$$

$$F_D^{(7)} \left(\begin{matrix} 1; \frac{1}{2} \\ 4 \end{matrix} \middle| z_1, \dots, z_7 \right) = \left(3 \left(\frac{1}{\sqrt{2}} - 1 \right) + \frac{3i}{\sqrt{2}} \right) \mathbf{K}(\sqrt{2}-1) \quad (4.27)$$

where

$$z_1 = 1 - \frac{1+i}{\sqrt{2}}, \quad z_2 = 1-i, \quad z_3 = 1 + \frac{1-i}{\sqrt{2}}, \quad z_4 = 2, \quad z_5 = 1 + \frac{1+i}{\sqrt{2}}, \quad z_6 = 1+i, \quad z_7 = 1 - \frac{1-i}{\sqrt{2}}$$

For (4.22) and (4.23) we use (3.10) obtaining

$$F_D^{(7)} \left(\begin{matrix} 5; \frac{1}{2} \\ \frac{11}{2} \end{matrix} \middle| \frac{1+i}{\sqrt{2}}, i, -\frac{1-i}{\sqrt{2}}, -1, -\frac{1+i}{\sqrt{2}}, -i, \frac{1-i}{\sqrt{2}} \right) = \frac{315\pi}{1024\sqrt{2}\mathbf{K}(\sqrt{2}-1)} \quad (4.28)$$

$$F_D^{(7)} \left(\begin{matrix} 7; \frac{1}{2} \\ \frac{15}{2} \end{matrix} \middle| \frac{1+i}{\sqrt{2}}, i, -\frac{1-i}{\sqrt{2}}, -1, -\frac{1+i}{\sqrt{2}}, -i, \frac{1-i}{\sqrt{2}} \right) = \frac{1001(2+\sqrt{2})\pi}{16384\mathbf{K}(\sqrt{2}-1)} \quad (4.29)$$

We add to our list of integrals one more, considered by Serret, [23] p. 65 and Legendre, [17] chapter XXX pp. 201-202, but (again) ignored by classical handbooks:

$$B_6(1, \frac{1}{3}) = \int_0^\infty \frac{dx}{\sqrt[3]{1+x^6}} = \frac{\sqrt[3]{4}}{\sqrt[4]{3}} \mathbf{K} \left(\frac{\sqrt{6}-\sqrt{2}}{4} \right) \quad (4.30)$$

Invoking again (3.18) to infer

$$F_D^{(6)} \left(\begin{matrix} 1; \frac{1}{3} \\ 2 \end{matrix} \middle| x_1^{(6)}, x_2^{(6)}, x_3^{(6)}, x_4^{(6)}, x_5^{(6)}, x_6^{(6)} \right) = \frac{\sqrt[3]{4}}{\sqrt[4]{3}} \mathbf{K} \left(\frac{\sqrt{6}-\sqrt{2}}{4} \right) \quad (4.31)$$

where

$$\begin{aligned} x_1^{(6)} &= \frac{\sqrt{3}}{2} + 1 + \frac{i}{2}, & x_2^{(6)} &= 1+i, & x_3^{(6)} &= 1 - \frac{\sqrt{3}}{2} + \frac{i}{2}, \\ x_4^{(6)} &= 1 - \frac{\sqrt{3}}{2} - \frac{i}{2}, & x_5^{(6)} &= 1-i, & x_6^{(6)} &= \frac{\sqrt{3}}{2} + 1 - \frac{i}{2} \end{aligned}$$

Eventually from (3.19) we get an evaluation in the analytic continuation of $F_D^{(5)}$. Putting

$$z_1^{(6)} = \frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad z_2^{(6)} = \frac{3}{2} - \frac{i\sqrt{3}}{2}, \quad z_3^{(6)} = 2, \quad z_4^{(6)} = \frac{3}{2} + \frac{i\sqrt{3}}{2}, \quad z_5^{(6)} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

we have

$$F_D^{(5)} \left(\begin{matrix} 1; \frac{1}{3} \\ 2 \end{matrix} \middle| z_1^{(6)}, z_2^{(6)}, z_3^{(6)}, z_4^{(6)}, z_5^{(6)} \right) = \frac{\sqrt[3]{4}}{\sqrt[4]{3}} \left(-\frac{\sqrt{3}}{2} + \frac{i}{2} \right) K \left(\frac{\sqrt{6} - \sqrt{2}}{4} \right) \quad (4.32)$$

In Chapter XXX section 164 pp. 205-206 of [17] Legendre evaluated an integral similar to (4.30)

$$A_6(1, \frac{1}{3}) = \int_0^1 \frac{dx}{\sqrt[3]{1-x^6}} = \frac{\sqrt[3]{4}}{\sqrt[4]{27}} K \left(\frac{\sqrt{6} - \sqrt{2}}{4} \right) \quad (4.33)$$

So using (3.10) we get the identity

$$F_D^{(5)} \left(\begin{matrix} 1; \frac{1}{3} \\ \frac{5}{3} \end{matrix} \middle| w_1^{(6)}, w_2^{(6)}, w_3^{(6)}, w_4^{(6)}, w_5^{(6)} \right) = \frac{\sqrt[3]{32}}{\sqrt[4]{2187}} K \left(\frac{\sqrt{6} - \sqrt{2}}{4} \right) \quad (4.34)$$

being

$$w_1^{(6)} = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad w_2^{(6)} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad w_3^{(6)} = -1, \quad w_4^{(6)} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad w_5^{(6)} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Furthermore in this special situation it is possible to use equation (2.4), i.e. Lemma 1.1 of [20], also for an integral of the A family, in (4.34) obtaining

$$F_D^{(4)} \left(\begin{matrix} 1; \frac{1}{3} \\ \frac{5}{3} \end{matrix} \middle| \frac{3}{2} + \frac{i\sqrt{3}}{2}, 1 + i\sqrt{3}, i\sqrt{3}, -\frac{1}{2} + \frac{i\sqrt{3}}{2} \right) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \frac{\sqrt[3]{32}}{\sqrt[4]{2187}} K \left(\frac{\sqrt{6} - \sqrt{2}}{4} \right) \quad (4.35)$$

5 Special values from reductions of hyperelliptic integrals

In a 1885 article [9], Goursat, using the change of variables:

$$x = \frac{t^3 + at + b}{3t - p}$$

obtained the reduction formula:

$$\begin{aligned} & \int \frac{dx}{\sqrt{x(4(3x-a)^3 - 27(b+px)^2)}} = \\ & \int \frac{-ap - 3b - 3t^2(p-2t)}{\sqrt{(at+b+t^3)(ap+3b+3t^2(p-2t))^2(4(ap+3b)+3pt^2+3t^3)}} dt \end{aligned} \quad (5.1)$$

Once a suitable interval of integration is chosen, it is possible to remove the sign uncertainty and simplify (5.1) obtaining

$$\int \frac{dx}{\sqrt{x(4(3x-a)^3 - 27(b+px)^2)}} = \pm \int \frac{dt}{\sqrt{(at+b+t^3)(4(ap+3b)+3pt^2+3t^3)}} \quad (5.2)$$

If we make the simple choice to take $a = p = 0$ and to integrate in $[(b^2/4)^{1/3}, \infty)$ in the left hand side of (5.2) we get

$$\int_{(b^2/4)^{1/3}}^{\infty} \frac{dx}{\sqrt{27x(4x^3 - b^2)}} = \int_{(b^2/4)^{1/3}}^{\infty} \frac{dt}{\sqrt{3(t^3+b)(t^3+4b)}} \quad (5.3)$$

Normalizing the interval of integration we see that the most natural choice $b = 2$ is not restrictive, so we can use, as starting point to find our next identity, the integrals equality:

$$\int_1^{\infty} \frac{dx}{\sqrt{x(x^3-1)}} = 6 \int_1^{\infty} \frac{dt}{\sqrt{(t^3+2)(t^3+8)}} \quad (5.4)$$

The hyperelliptic integral at the right hand side of (5.4) is a particular case of the family of integrals described by the following Lemma.

Lemma 5.1. *Let $2n - m > 0$, $a, b > 0$ then*

$$\int_1^\infty \frac{dt}{\sqrt[n]{(t^n + a)(t^n + b)}} = \frac{m}{2n - m} F_1 \left(\begin{matrix} \frac{2n-m}{mn}; \frac{1}{m}, \frac{1}{m} \\ \frac{2n-m+mn}{mn} \end{matrix} \middle| -a, -b \right) \quad (5.5)$$

Proof. Condition $2n - m > 0$ ensures the convergence of the integral at the right hand side of (5.5). Thesis follows immediately from the integral representation theorem for the Appell F_1 after the change of variables $t = 1/u^{1/n}$. \square

From (5.4) and Lemma 5.1 we infer the following new evaluation in the analytic continuation of F_1 .

Theorem 5.2.

$$F_1 \left(\begin{matrix} \frac{2}{3}; \frac{1}{2}, \frac{1}{2} \\ \frac{5}{3} \end{matrix} \middle| -2, -8 \right) = \frac{1}{\pi \sqrt[3]{16}\sqrt{3}} \Gamma^3(1/3). \quad (5.6)$$

Proof. Using Lemma 5.1 in equation (5.4) we see that

$$\int_1^\infty \frac{dx}{\sqrt{x(x^3 - 1)}} = 3F_1 \left(\begin{matrix} \frac{2}{3}; \frac{1}{2}, \frac{1}{2} \\ \frac{5}{3} \end{matrix} \middle| -2, -8 \right) \quad (5.7)$$

But we also have

$$\int_1^\infty \frac{dx}{\sqrt{x(x^3 - 1)}} = \int_0^1 \frac{dx}{\sqrt{1 - x^3}} = \frac{\sqrt{\pi}}{3} \frac{\Gamma(1/3)}{\Gamma(5/6)} \quad (5.8)$$

Thesis (5.6) follows by equating (5.7) and (5.8) and using Euler reflection formula. \square

Remark 5.3. If, instead of using the Euler function to evaluate the integral in (5.8), we employ the elliptic integral of first kind, using [6] entry 244.00 p. 92 we get:

$$F_1 \left(\begin{matrix} \frac{2}{3}; \frac{1}{2}, \frac{1}{2} \\ \frac{5}{3} \end{matrix} \middle| -2, -8 \right) = \frac{1}{3^{5/4}} F \left(\arccos(2 - \sqrt{3}), \frac{\sqrt{6} + \sqrt{2}}{4} \right). \quad (5.6a)$$

In order to express the Appell F_1 not in its analytic continuation, but in the domain of convergence of the double power series which defines it, we present a slight modification of the Goursat reduction formula.

$$\int_0^1 \frac{dx}{\sqrt{1 - x^3}} = \int_0^1 \frac{6}{\sqrt{(t^3 + 2)(t^3 + 8)}} dt. \quad (5.9)$$

Identity (5.9) comes from the change of variable

$$x = \frac{3t}{t^3 + 2}.$$

The hyperelliptic integral at the right hand side of (5.9) is computable using the Appell F_1 hypergeometric function of two variables. In fact we have, in analogy with Lemma 5.1:

Lemma 5.4. *Let $a, b > 0$ then*

$$\int_0^1 \frac{dt}{\sqrt[n]{(t^n + a)(t^n + b)}} = \frac{1}{\sqrt[n]{ab}} F_1 \left(\begin{matrix} \frac{1}{n}; \frac{1}{m}, \frac{1}{m} \\ \frac{1}{n} + 1 \end{matrix} \middle| -\frac{1}{a}, -\frac{1}{b} \right) \quad (5.10)$$

From Lemma 5.4 we get

$$\int_0^1 \frac{dt}{\sqrt{(t^3 + 2)(t^3 + 8)}} = \frac{1}{4} F_1 \left(\begin{matrix} \frac{1}{3}; \frac{1}{2}, \frac{1}{2} \\ \frac{4}{3} \end{matrix} \middle| -\frac{1}{2}, -\frac{1}{8} \right). \quad (5.11)$$

We can state a further identity involving π which concerns a new evaluation of the Appell F_1 .

Theorem 5.5.

$$F_1 \left(\begin{matrix} \frac{1}{3}; \frac{1}{2}, \frac{1}{2} \\ \frac{4}{3} \end{matrix} \middle| -\frac{1}{2}, -\frac{1}{8} \right) = \frac{1}{\pi \sqrt{27} \sqrt[3]{2}} \Gamma^3 \left(\frac{1}{3} \right). \quad (5.12)$$

Proof. To get the thesis in this situation we evaluate the elliptic integral at the left hand side of (5.9) using the Euler Beta function:

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\sqrt{\pi}}{3} \frac{\Gamma(1/3)}{\Gamma(5/6)}. \quad (5.13)$$

Thesis (5.12) follows from (5.9) evaluating the integrals therein using (5.11) (right hand side) and (5.13) (left hand side) and again invoking Euler reflection formula. \square

Remark 5.6. If, instead of using the Euler Beta, we evaluate the elliptic integral at the left hand side of (5.9) using entry 244.00 p. 92 of [6] we get, instead of (5.13):

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F \left(\arccos(2 - \sqrt{3}), \frac{\sqrt{6} + \sqrt{2}}{4} \right) \quad (5.14)$$

leading to the elliptic analogue of (5.12):

$$F_1 \left(\begin{matrix} \frac{1}{3}; \frac{1}{2}, \frac{1}{2} \\ \frac{4}{3} \end{matrix} \middle| -\frac{1}{2}, -\frac{1}{8} \right) = \frac{2}{3\sqrt[4]{3}} F \left(\arccos(2 - \sqrt{3}), \frac{\sqrt{6} + \sqrt{2}}{4} \right) \quad (5.15)$$

The family of integral identities pointed by the Goursat transformation is huge. Nevertheless the evaluation is intricate when some of the cubics in (5.2) are complete. Taking for instance $a = 0$, $b = p = 1$ in (5.2) we get the identity

$$\int_1^\infty \frac{dx}{\sqrt{x(x-1)(4x^2+3x+1)}} = \int_1^\infty \frac{3}{\sqrt{(t^3+1)(t^3+t^2+4)}} dt \quad (5.16)$$

which can be rewritten as

$$\int_0^1 \frac{dx}{\sqrt{(1-x)(x^2+3x+4)}} = \int_0^1 \frac{3t}{\sqrt{(t^3+1)(4t^3+t+1)}} dt. \quad (5.17)$$

We need to employ the Lauricella $F_D^{(6)}$ in order to evaluate the hyperelliptic integral at right hand side of (5.17). To do it, first observe that:

$$\begin{aligned} t^3 + 1 &= (1+t) \left(1 - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2} \right) t \right) \left(1 - \left(\frac{1}{2} + \frac{i\sqrt{3}}{2} \right) t \right) = (1+t)(1-\omega t)(1-\bar{\omega}t) \\ 4t^3 + t + 1 &= (1+2t) \left(1 - \left(\frac{1}{2} - \frac{i\sqrt{7}}{2} \right) t \right) \left(1 - \left(\frac{1}{2} + \frac{i\sqrt{7}}{2} \right) t \right) = (1+2t)(1-\varepsilon t)(1-\bar{\varepsilon}t) \end{aligned}$$

hence

$$\int_0^1 \frac{t}{\sqrt{(t^3+1)(4t^3+t+1)}} dt = \frac{1}{2} F_D^{(6)} \left(\begin{matrix} 2; \frac{1}{2} \\ 3 \end{matrix} \middle| -1, \omega, \bar{\omega}, -2, \varepsilon, \bar{\varepsilon} \right) \quad (5.16c)$$

We follow the same strategy to evaluate the integral at the left hand side of (5.17) starting from the factorization

$$x^2 + 3x + 4 = 4 \left(1 - \left(-\frac{3}{8} - \frac{i\sqrt{7}}{8} \right) x \right) \left(1 - \left(-\frac{3}{8} + \frac{i\sqrt{7}}{8} \right) x \right) = 4(1-\alpha x)(1-\bar{\alpha}x)$$

leading to the integration formula:

$$\int_0^1 \frac{dx}{\sqrt{(1-x)(x^2+3x+4)}} = F_1 \left(\begin{matrix} 1; \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix} \middle| \alpha, \bar{\alpha} \right). \quad (5.16d)$$

So we proved the following hypergeometric reduction formula

Theorem 5.7. *Defining:*

$$\alpha = -\frac{3}{8} - \frac{i\sqrt{7}}{8}, \quad \omega = \frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad \varepsilon = \frac{1}{2} - \frac{i\sqrt{7}}{2}$$

Then we have

$$F_D^{(6)} \left(\begin{matrix} 2; \frac{1}{2} \\ 3 \end{matrix} \middle| -1, \omega, \bar{\omega}, -2, \varepsilon, \bar{\varepsilon} \right) = \frac{2}{3} F_1 \left(\begin{matrix} 1; \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix} \middle| \alpha, \bar{\alpha} \right). \quad (5.18)$$

Remark 5.8. Of course, in evaluating the elliptic integral in (5.16d) we can follow the elliptic way, [6] entry 243.00 p. 91:

$$\int_0^1 \frac{dx}{\sqrt{(1-x)(x^2+3x+4)}} = \frac{1}{2^{3/4}} F \left(\arccos \left(\frac{1}{7} (9 - 4\sqrt{2}) \right), \frac{1}{4} \sqrt{8 + 5\sqrt{2}} \right). \quad (5.19)$$

So, comparing (5.17) and (5.19) we get

$$F_D^{(6)} \left(\begin{matrix} 2; \frac{1}{2} \\ 3 \end{matrix} \middle| -1, \omega, \bar{\omega}, -2, \varepsilon, \bar{\varepsilon} \right) = \frac{\sqrt[4]{2}}{3} F \left(\arccos \left(\frac{1}{7} (9 - 4\sqrt{2}) \right), \frac{1}{4} \sqrt{8 + 5\sqrt{2}} \right). \quad (5.20)$$

Remark 5.9. Observe that the Lauricella $F_D^{(6)}$ appearing in (5.18) and (5.20) fulfills hypotheses of Lemma 1.1 of [20]; thus in (5.18) and (5.20) the left hand side can be replaced with a lower order Lauricella.

$$\frac{1}{4} F_D^{(5)} \left(\begin{matrix} 2; \frac{1}{2} \\ 3 \end{matrix} \middle| -\frac{1}{2}, \frac{3-i\sqrt{3}}{4}, \frac{3+i\sqrt{3}}{4}, \frac{3-i\sqrt{7}}{4}, \frac{3+i\sqrt{7}}{4} \right)$$

6 Conclusions

The paper holds the proofs of a set of formulae providing the values in some special points -either inside or outside the unity disk- of some high order hypergeometric functions. Among them we can conclusively cite e. g. the following evaluations. First the (3.15) providing the Lauricella of order $2m$; (4.27) and the (4.28) for Lauricella of index 7 through complete elliptic integral of first kind; the (5.6) providing the Appell function through values of Gamma; equation (5.15) provides an Appell function again through incomplete elliptic integrals of first kind; and so on. They have been theoretically obtained either by the *double evaluation method* or working on some established formulae for the reduction of hyperelliptic integrals.

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